A Geometric Application of Nori’s Connectivity Theorem

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Abstract. We study (rational) sweeping out of general hypersurfaces by varieties having small moduli spaces.
As a consequence, we show that general $K$-trivial hypersurfaces are not rationally swept out by abelian varieties of dimension at least two.
As a corollary, we show that Clemens’ conjecture on the finiteness of rational curves of given degree in a general quintic threefold, and Lang’s conjecture saying that such varieties should be rationally swept-out by abelian varieties, contradict.

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0. – Introduction

Our purpose in this paper is to contribute to the study of rational maps from $r$-dimensional varieties to general hypersurfaces in projective space (cf [5], [20], [12], [4]). In the last section, we shall eventually extend this to the study of correspondences instead of rational maps. The problem we consider is the following: given a family $\mathcal{Y} \to S$ of $r$-dimensional smooth projective varieties, when is a general hypersurface $X$ of degree $d$ in projective space $\mathbb{P}^{n+1}$ swept out by images of rational maps from one member of this family to $X$?

(Recall that the word “general” in this context means “away from countably many proper Zariski closed subsets of the moduli space”.)

Our approach to this problem is Hodge theoretic. Unlike [5], [20], [12], [4], the result has nothing to do with the canonical bundle of the varieties $Y_t$, $t \in S$. Instead, our answer will depend only on the dimension of the moduli space $S$. Roughly speaking, the idea is as follows: assume that dim $S$ is small, but the general $X$ is covered by images of rational maps from one member of this family to $X$; then there is a universal dominating rational map $\Phi$ fitting in the following commutative diagram

$$
\Phi : \mathcal{X} \longrightarrow X_U \\
\downarrow \pi \downarrow \\
B \rightarrow U,
$$