The permutation group method for the dilogarithm

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Abstract. We give qualitative and quantitative improvements on all the best previously known irrationality results for dilogarithms of positive rational numbers. We obtain such improvements by applying our permutation group method to the diophantine study of double integrals of rational functions related to the dilogarithm.

Mathematics Subject Classification (2000): 11J82 (primary); 33B30, 20B35 (secondary).

1. Introduction

1.1 For $k \geq 1$ integer, the polylogarithm $\text{Li}_k(x)$ of order $k$ is defined, in the unit disc $|x| < 1$, by the power series

$$\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$ 

In particular

$$\text{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$$

and

$$\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t} \, dt.$$ 

The polylogarithm, which arises in Euler’s work, is a typical instance of $G$-function. Following Siegel ([6], erster Teil, Section 4, VII), a $G$-function is defined to be a Taylor series $\sum a_n x^n$ whose coefficients are algebraic numbers such that, for a positive constant $C$ independent of $n$, (i) $a_n$ and its conjugates do not exceed $C^n$, and (ii) there is a common denominator for the coefficients $a_v$ with $v \leq n$ which does not exceed $C^n$.

Pervenuto alla Redazione il 4 novembre 2004 e in forma definitiva il 27 luglio 2005.