**Riesz transform on manifolds and Poincaré inequalities**

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**Abstract.** We study the validity of the $L^p$ inequality for the Riesz transform when $p > 2$ and of its reverse inequality when $1 < p < 2$ on complete Riemannian manifolds under the doubling property and some Poincaré inequalities.

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**Introduction**

Let $M$ be a non-compact complete Riemannian manifold. Denote by $\mu$ the Riemannian measure, and by $\nabla$ the Riemannian gradient. Denote by $|.|$ the length in the tangent space, and by $\| . \|_p$ the norm in $L^p(M, \mu)$, $1 \leq p \leq \infty$. One defines $\Delta$, the Laplace-Beltrami operator, as a self-adjoint positive operator on $L^2(M, \mu)$ by the formal integration by parts

$$(\Delta f, f) = \| |\nabla f| \|_2^2$$

for all $f \in C_0^\infty(M)$, and its positive self-adjoint square root $\Delta^{1/2}$ by

$$(\Delta f, f) = \| \Delta^{1/2} f \|_2^2.$$  

As a consequence,

$$\| |\nabla f| \|_2^2 = \| \Delta^{1/2} f \|_2^2.$$ \hfill (E2)

To identify the spaces defined by (completion with respect to) the seminorms $\| |\nabla f| \|_p$ and $\| \Delta^{1/2} f \|_p$ on $C_0^\infty(M)$ for some $p \in (1, \infty)$, it is enough to prove that there exist $0 < c_p \leq C_p < \infty$ such that for all $f \in C_0^\infty(M)$

$$c_p \| \Delta^{1/2} f \|_p \leq \| |\nabla f| \|_p \leq C_p \| \Delta^{1/2} f \|_p.$$ \hfill (E_p)

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