Geometric rigidity of conformal matrices

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Abstract. We provide a geometric rigidity estimate à la Friesecke-James-Müller for conformal matrices. Namely, we replace SO\(_n\) by an arbitrary compact set of conformal matrices, bounded away from 0 and invariant under SO\(_n\), and rigid motions by Möbius transformations.

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1. Introduction

This paper is concerned with the so-called geometric rigidity estimates for conformal matrices. Recently, Friesecke, James and Müller developed a successful new approach to the classical problem of dimension-reduction in nonlinear elasticity [7, 8]. A fundamental ingredient was the following rigidity estimate for the group SO\(_n\) = \{A ∈ \(M^{n×n}\) : \(A^TA = I\), det\(A = 1\)\} of special orthogonal matrices in \(\mathbb{R}^n\).

Theorem 1.1. Let \(Ω \subset \mathbb{R}^n\) be a bounded Lipschitz domain and \(n \geq 2\). There exists a constant \(C_1 = C_1(Ω)\) with the property that for each \(v \in W^{1,2}(Ω, \mathbb{R}^n)\), there exists \(R \in SO(n)\) such that

\[
\|Dv - R\|_{L^2(Ω)} \leq C_1 \|\text{dist}_{SO(n)}(Dv)\|_{L^2(Ω)}.
\]  

(1.1)

Theorem 1.1 has been used in a number of related problems concerning dimension-reduction, e.g. [3, 6, 18] and [17]. In all the applications it is crucial that the dependence between the left- and right-hand side is linear and that \(v\) is any general Sobolev mapping (the classical result of John [15] gives an \(L^2 - L^∞\) estimate valid for locally bi-Lipschitz maps). Theorem 1.1 makes quantitative the following classical result of Reshetnyak [19] for sequences.

Theorem 1.2. Let \(\{v_j\} \in W^{1,2}(Ω)\) be a weakly convergent sequence in \(W^{1,2}\). Then there exists \(R \in SO(n)\) such that

\[
\lim_{j \to \infty} \|\text{dist}_{SO(n)}(Dv_j)\|_{L^2(Ω)} = 0 \Rightarrow \lim_{j \to \infty} \|Dv_j - R\|_{L^2(Ω)} = 0.
\]  

(1.2)

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