On the absence of the one-sided Poincaré lemma in Cauchy-Riemann manifolds

FABIO NICOLA

Abstract. Given an embeddable CR manifold $M$ and a non-characteristic hypersurface $S \subset M$ we present a necessary condition for the tangential Cauchy-Riemann operator $\overline{\partial}_M$ on $M$ to be locally solvable near a point $x_0 \in S$ in one of the sides determined by $S$.

Mathematics Subject Classification (2000): 32W10 (primary); 58J10 (secondary).

1. Introduction and discussion of the results

Let $M$ be a CR manifold of type $(n, d)$ (so that $\dim M = 2n + d$). For an open subset $\mathcal{O} \subset M$ we denote by $C^\infty(\mathcal{O}, \Lambda^{p,q})$ the space of smooth $(p, q)$ forms in $\mathcal{O}$, $0 \leq p \leq m$, $0 \leq q \leq n$, $m = n + d$ (see e.g. Treves [16] and Section 2 below for terminology).

The Poincaré lemma is said to hold for the tangential Cauchy-Riemann complex $\overline{\partial}_M$ in degree $q$, $1 \leq q \leq n$, at the point $x_0 \in M$, if for every open neighborhood $\Omega$ of $x_0$ there is an open neighborhood $\Omega' \subset \Omega$ such that the system

$$\overline{\partial}_M u = f$$

admits a solution $u \in C^\infty(\Omega', \Lambda^{0,q-1})$ for all $f \in C^\infty(\Omega, \Lambda^{0,q})$ which are cocycles, i.e. $\overline{\partial}_M$-closed (indeed we have $\overline{\partial}_M^2 = 0$, so that this condition is necessary). By a classical argument due to Grothendieck the definition given is in fact equivalent to the apparently weaker version of solvability in the sense of germs of smooth forms.

Necessary and sufficient conditions for the Poincaré lemma to hold have been object of investigation by many authors; we refer the reader, among others, to the important contributions by Lewy [10], Hörmander [8], Andreotti and Hill [2], Andreotti, Fredricks and Nacinovich [1], Folland and Stein [6], Nacinovich [13, 14], Michel [12], Treves [16, 18], Chen and Shaw [5], Hill and Nacinovich [7], Peloso

Pervenuto alla Redazione il 30 giugno 2005 e in forma definitiva il 12 ottobre 2005.