A quantitative version of the isoperimetric inequality:  
the anisotropic case

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Abstract. We state and prove a stability result for the anisotropic version of the isoperimetric inequality. Namely if \( E \) is a set with small anisotropic isoperimetric deficit, then \( E \) is “close” to the Wulff shape set.

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1. Introduction and main results

Let \( \Gamma : \mathbb{R}^N \to [0, +\infty) \) be a positively 1–homogeneous convex function such that \( \Gamma(x) > 0 \) for all \( x \neq 0 \). The Wulff problem associated to \( \Gamma \) is

\[
\text{Min} \left\{ \int_{\partial^* E} \Gamma(v^E(x)) \, d\mathcal{H}^{N-1} : \mathcal{L}^N(E) = \text{const} \right\},
\]

where \( E \) ranges among all sets of finite perimeter satisfying the constraint \( \mathcal{L}^N(E) = \text{const} \) Here \( v^E \) is the (generalized) outer normal to \( E \) and \( \partial^* E \) is the (reduced) boundary of \( E \) (which equals the usual boundary \( \partial E \) if \( E \) is smooth). For an anisotropic function \( \Gamma \), one of the first attempts to solve this problem is contained in a paper by G. Wulff [22] dating back to 1901. However, it was only in 1944 that A. Dinghas [9] proved that within the special class of convex polytopes the minimiser of (1.1) is a set homothetic to the unit ball of the dual norm of \( \Gamma(x) \), i.e.,

\[
W_\Gamma = \{ x \in \mathbb{R}^N : \langle x, v \rangle - \Gamma(v) < 0 \text{ for all } v \in \mathbb{S}^{N-1} \},
\]

which is known as the Wulff shape set.

Introducing the quantity

\[
P_\Gamma(E) = \int_{\partial^* E} \Gamma(v^E(x)) \, d\mathcal{H}^{N-1},
\]