The $BV$-energy of maps into a manifold: relaxation and density results

MARIANO GIAQUINTA AND DOMENICO MUCCI

Abstract. Let $\mathcal{Y}$ be a smooth compact oriented Riemannian manifold without boundary, and assume that its 1-homology group has no torsion. Weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded total variation give rise to equivalence classes of Cartesian currents in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ for which we introduce a natural $BV$-energy. Assume moreover that the first homotopy group of $\mathcal{Y}$ is commutative. In any dimension $n$ we prove that every element $T$ in $\text{cart}^{1,1}(B^n \times \mathcal{Y})$ can be approximated weakly in the sense of currents by a sequence of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with total variation converging to the $BV$-energy of $T$. As a consequence, we characterize the lower semicontinuous envelope of functions of bounded variations from $B^n$ into $\mathcal{Y}$.

Mathematics Subject Classification (2000): 49Q15 (primary); 49Q20 (secondary).

In this paper we deal with sequences of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded total variation

$$\sup_k \mathcal{E}_{1,1}(u_k) < \infty, \quad \mathcal{E}_{1,1}(u_k) := \int_{B^n} |Du_k| \, dx$$

and their limit points. Here $B^n$ is the unit ball in $\mathbb{R}^n$ and $\mathcal{Y}$ is a smooth oriented Riemannian manifold of dimension $M \geq 1$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 2$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary. In addition, we assume that the integral 1-homology group $H_1(\mathcal{Y}) := H_1(\mathcal{Y}; \mathbb{Z})$ has no torsion.

Modulo passing to a subsequence the $(n,1)$-currents $G_{u_k}$, integration over the graphs of $u_k$ of $n$-forms with at most one vertical differential, converge to a current $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$, see Section 2 below. To every $T \in \text{cart}^{1,1}(B^n \times \mathcal{Y})$ it corresponds a function $u_T \in BV(B^n, \mathcal{Y})$, i.e., $u_T \in BV(B^n, \mathbb{R}^N)$ such that $u_T(x) \in \mathcal{Y}$ for $\mathcal{L}^n$-a.e. $x \in B^n$, compare [14, Vol. I, Section 4.2] [14, Vol. II, Section 5.4]. Also, the weak convergence $G_{u_k} \rightharpoonup T$ yields the convergence $u_k \rightharpoonup u_T$ weakly in the $BV$-sense.

Received June 12, 2006; accepted in revised form October 17, 2006.