The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions

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Abstract. We first discuss a class of inequalities of Onofri type depending on a parameter, in the two-dimensional Euclidean space. The inequality holds for radial functions if the parameter is larger than $-1$. Without symmetry assumption, it holds if and only if the parameter is in the interval $(-1, 0]$. The inequality gives us some insight on the symmetry breaking phenomenon for the extremal functions of the Caffarelli-Kohn-Nirenberg inequality, in two space dimensions. In fact, for suitable sets of parameters (asymptotically sharp) we prove symmetry or symmetry breaking by means of a blow-up method and a careful analysis of the convergence to a solution of a Liouville equation. In this way, the Onofri inequality appears as a limit case of the Caffarelli-Kohn-Nirenberg inequality.

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1. Introduction

The Onofri inequality on the sphere $S^2$, see for instance [1, 14, 15], states that

$$\int_{S^2} e^{2u - \int_{S^2} u \, d\sigma} \, d\sigma \leq e^{\|\nabla u\|_{L^2(S^2)}^2}$$  \hspace{1cm} (1.1)

for all $u \in \mathcal{E} = \{ u \in L^1(S^2, d\sigma) : |\nabla u| \in L^2(S^2, d\sigma) \}$, where $d\sigma$ denotes the measure induced by Lebesgue’s measure in $\mathbb{R}^3 \supset S^2$, normalized so that $\int_{S^2} d\sigma = 1$. Using the stereographic projection from $S^2$ onto $\mathbb{R}^2$, we see that (1.1) is equivalent to the following Onofri type inequality in $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} e^{u - \int_{\mathbb{R}^2} u \, d\mu} \, d\mu \leq e^{\frac{1}{\pi^2} \|\nabla u\|_{L^2(\mathbb{R}^2)}^2}$$

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