Convex isoperimetric sets in the Heisenberg group

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Abstract. We characterize convex isoperimetric sets in the Heisenberg group. We first prove Sobolev regularity for a certain class of $\mathbb{R}^2$-valued vector fields of bounded variation in the plane related to the curvature equations. Then we show that the boundary of convex isoperimetric sets is foliated by geodesics of the Carnot-Carathéodory distance.

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1. Introduction

We identify the Heisenberg group $\mathbb{H}^1$ with $\mathbb{C} \times \mathbb{R}$ endowed with the group law

\[(z, t)(z', t') = (z + z', t + t' + 2 \text{Im}(z\overline{z}')),\]

where $t, t' \in \mathbb{R}$ and $z = x + iy, z' = x' + iy' \in \mathbb{C}$. The Lie algebra of left-invariant vector fields is spanned by

\[X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t} \quad \text{and} \quad T = \frac{\partial}{\partial t},\]

and the distribution of planes spanned by $X$ and $Y$, called horizontal distribution, generates the Lie algebra by brackets.

The natural volume in $\mathbb{H}^1$ is the Haar measure, which, up to a positive factor, coincides with Lebesgue measure in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. Lebesgue measure is also the Riemannian volume of the left-invariant metric for which $X, Y$ and $T$ are orthonormal. We denote by $|E|$ the volume of a (Lebesgue) measurable set $E \subset \mathbb{H}^1$. The horizontal perimeter (or simply perimeter) of $E$ is

\[P(E) = \sup \left\{ \int_E (X\varphi_1 + Y\varphi_2) \, dx dy dt \bigg| \varphi_1, \varphi_2 \in C_c^1(\mathbb{R}^3), \ \varphi_1^2 + \varphi_2^2 \leq 1 \right\}. \quad (1.1)\]

If $P(E) < +\infty$, the set $E$ is said to be of finite perimeter. Perimeter is left-invariant and 3-homogeneous with respect to the group of dilations $\delta_\lambda : \mathbb{H}^1 \rightarrow \mathbb{H}^1, \delta_\lambda(z, t) =$

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