On the stability of the universal quotient bundle restricted to congruences of low degree of $\mathbb{G}(1, 3)$

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Abstract. We study the semistability of $Q|_S$, the universal quotient bundle on $\mathbb{G}(1,3)$ restricted to any smooth surface $S$ (called congruence). Specifically, we deduce geometric conditions for a congruence $S$, depending on the slope of a saturated linear subsheaf of $Q|_S$. Moreover, we check that the Dolgachev-Reider Conjecture (i.e. the semistability of $Q|_S$ for nondegenerate congruences $S$) is true for all the congruences of degree less than or equal to 10. Also, when the degree of a congruence $S$ is less than or equal to 9, we compute the highest slope reached by the linear subsheaves of $Q|_S$.

Mathematics Subject Classification (2010): 14J60 (primary); 14M07, 14M15 (secondary).

Introduction

The main numerical invariant of a congruence, i.e. a smooth irreducible surface of the Grassmann variety $\mathbb{G}(1, 3)$ of lines in $\mathbb{P}^3$, is its bidegree $(a, b)$. Regarding a congruence as a two-dimensional family of lines in $\mathbb{P}^3$, the order $a$ is defined to be the number of lines of the family passing through a general point of $\mathbb{P}^3$; analogously, the class $b$ is the number of lines of the family contained in a general plane of $\mathbb{P}^3$. An interesting problem that arises in a natural way is the following: given two integers $a$ and $b$, does there exist a congruence of bidegree $(a, b)$?

About this question, the best known result is the bound $a \leq O\left(\frac{b^4}{3}\right)$ for every congruence of bidegree $(a, b)$, given by Gross in [8] (although there are not known examples for which $|a - b| > O\left(\frac{b^{3/4}}{3}\right)$). A new approach was introduced by Dolgachev and Reider by means of vector bundles. More precisely, they stated in [7] the following conjecture:

This paper has been written in the framework of the research projects MTM2006-04785 (funded by the Spanish Ministry of Education) and CCG07-UCM/ESP-3026 (funded by University Complutense and the regional government of Madrid).

Received February 11, 2009; accepted in revised form July 24, 2009.