Forward, backward and elliptic Harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations

EMMANUELE DiBENEDETTO, UGO GIANAZZA AND VINCENZO VESPRI

Abstract. Forward, backward and elliptic Harnack inequalities for non-negative solutions of a class of singular, quasi-linear, parabolic equations, are established. These classes of singular equations include the $p$-Laplacean equation and equations of the porous medium type. Key novel points include form of a Harnack estimate backward in time, that has never been observed before, and measure theoretical proofs, as opposed to comparison principles. These Harnack estimates are established in the super-critical range (1.5) below. Such a range is optimal for a Harnack estimate to hold.

Mathematics Subject Classification (2010): 35K65 (primary); 35B65, 35B45 (secondary).

1. Main results

Let $E$ be an open set in $\mathbb{R}^N$ and for $T > 0$ let $E_T = E \times (0, T]$. Let $u$ be a weak solution

$$u \in C_{\text{loc}}(0, T; L^2_{\text{loc}}(E)) \cap L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(E)) \quad 1 < p < 2$$

(1.1)

of a quasi-linear, singular parabolic equation of the type

$$u_t - \text{div} A(x, t, u, Du) = B(x, t, u, Du) \quad \text{weakly in } E_T$$

(1.2)

where the functions $A : E_T \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases}
A(x, t, u, Du) \cdot Du \geq C_o |Du|^p - C^p \\
|A(x, t, u, Du)| \leq C_1 |Du|^{p-1} + C^{p-1} \quad \text{a.e. in } E_T \\
|B(x, t, u, Du)| \leq C |Du|^{p-1} + C^p
\end{cases}$$

(1.3)

This work has been partially supported by I.M.A.T.I. – C.N.R. – Pavia.
DiBenedetto’s work partially supported by National Science Foundation grant DMS-0652385.
Received March 9, 2009; accepted in revised form May 31, 2009.