On the de Rham cohomology of solvmanifolds

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Abstract. Using results by D. Witte [35] on the superrigidity of lattices in solvable Lie groups we get a new proof of a recent remarkable result obtained by D. Guan [15] on the de Rham cohomology of a compact solvmanifold, i.e., of a quotient of a connected and simply connected solvable Lie group $G$ by a lattice $\Gamma$. This result can be applied to compute the Betti numbers of a compact solvmanifold $G/\Gamma$ even in the case that the solvable Lie group $G$ and the lattice $\Gamma$ do not satisfy the Mostow condition.

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1. Introduction

Let $M$ be a compact solvmanifold, i.e., a quotient of a connected and simply connected solvable Lie group $G$ by a lattice $\Gamma$. Denote by $\text{Ad}_G(G)$ (respectively, $\text{Ad}_G(\Gamma)$) the subgroup of $\text{GL}(g)$ generated by $e^{\text{ad}X}$, for all $X$ in the Lie algebra $g$ of $G$ (respectively, in the Lie algebra of $\Gamma$). It is well known that if $G$ is a simply connected solvable Lie group, then $\text{Ad}_G(G)$ is a solvable algebraic group and $\text{Aut}(G) \cong \text{Aut}(g)$. We will denote by $\mathcal{A}(\text{Ad}_G(G))$ and $\mathcal{A}(\text{Ad}_G(\Gamma))$ the real algebraic closures of $\text{Ad}_G(G)$ and $\text{Ad}_G(\Gamma)$ respectively.

In general, as a consequence of the Borel density theorem (see [34, Corollary 4.2] and Theorem 3.1 here) applied to the adjoint representation, one has that if $\Gamma$ is a lattice of a connected solvable Lie group $G$, then $\mathcal{A}(\text{Ad}_G(G)) = T_{\text{cpt}} \mathcal{A}(\text{Ad}_G(\Gamma))$ is a product of the groups $T_{\text{cpt}}$ and $\mathcal{A}(\text{Ad}_G(\Gamma))$, where $T_{\text{cpt}}$ is any maximal compact torus of $\mathcal{A}(\text{Ad}_G(G))$.

If $\mathcal{A}(\text{Ad}_G(G)) = \mathcal{A}(\text{Ad}_G(\Gamma))$, i.e., if $G$ and $\Gamma$ satisfy the Mostow condition, then the de Rham cohomology $H_{\text{dR}}^*(M)$ of the compact solvmanifold $M = G/\Gamma$ can be computed by the Chevalley-Eilenberg cohomology $H^*(g)$ of the Lie algebra $g$ of $G$ (see [26] and [30, Corollary 7.29]); indeed, one has the isomorphism

$$H_{\text{dR}}^*(M) \cong H^*(g).$$

(1.1)

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