Two solutions for a singular elliptic equation
by variational methods

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Abstract. We find two nontrivial solutions of the equation
\(-\Delta u = (-\frac{1}{u^\beta} + \lambda u^p)\chi_{\{u>0\}}\) in \(\Omega\) with
Dirichlet boundary condition, where \(0 < \beta < 1\) and \(0 < p < 1\). In the first
approach we consider a sequence of \(\varepsilon\)-problems with \(1/\mu^\beta\)
replaced by \(u^q/(u + \varepsilon)^{q+\beta}\) with \(0 < q < p < 1\). When the parameter
\(\lambda > 0\) is large enough, we find two critical points of the corresponding
\(\varepsilon\)-functional which, at the limit as \(\varepsilon \to 0\), give rise to two distinct
nonnegative solutions of the original problem. Another approach is based on
perturbations of the domain \(\Omega\), we then find a unique positive solution
for \(\lambda\) large enough. We derive gradient estimates to guarantee convergence of
approximate solutions \(u_\varepsilon\) to a true solution \(u\) of the problem.

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1. Introduction

In this paper we prove that the problem

\[
\begin{aligned}
-\Delta u &= \left(-\frac{1}{u^\beta} + \lambda u^p\right)\chi_{\{u>0\}} & \text{in } \Omega \\
 u &= 0 & \text{on } \partial \Omega
\end{aligned}
\]  

(1.1)

has two nonnegative solutions when the parameter \(\lambda > 0\) is large. The expression
\(\chi_{\{u>0\}}\) denotes the characteristic function corresponding to the set \(\{u > 0\}\). Here-
after, \(\Omega \subset \mathbb{R}^N\), \(N \geq 1\), is a bounded domain, \(0 < \beta < 1\) and \(0 < p < 1\). By a
solution we mean a function \(u \in H^1_0(\Omega)\) satisfying (1.1) in the weak sense, that is,

\[
\int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega \cap \{u>0\}} \left(-\frac{1}{u^\beta} + \lambda u^p\right) \varphi
\]

for every \(\varphi \in C^1_c(\Omega)\).

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