On fundamental groups related to degeneratable surfaces: conjectures and examples

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Abstract. We argue that for a smooth surface $S$, considered as a ramified cover over $\mathbb{CP}^2$, branched over a nodal-cuspidal curve $B \subset \mathbb{CP}^2$, one could use the structure of the fundamental group of the complement of the branch curve $\pi_1(\mathbb{CP}^2 - B)$ to understand other properties of the surface and its degeneration and vice-versa. In this paper, we look at embedded-degeneratable surfaces — a class of surfaces admitting a planar degeneration with a few combinatorial conditions imposed on its degeneration. We close a conjecture of Teicher on the virtual solvability of $\pi_1(\mathbb{CP}^2 - B)$ for these surfaces and present two new conjectures on the structure of this group, regarding non-embedded-degeneratable surfaces. We prove two theorems supporting our conjectures, and show that for $\mathbb{CP}^1 \times C_g$, where $C_g$ is a curve of genus $g$, $\pi_1(\mathbb{CP}^2 - B)$ is a quotient of an Artin group associated to the degeneration.

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1. Introduction

Given a smooth algebraic projective variety $X$, one of the main techniques used to obtain information on $X$ is to degenerate it to a union of “simpler” varieties. The “simplest” degeneration can be thought as the degeneration of $X$ to a union of dim $X$-planes, and one would like to use the combinatorial data induced from this arrangement of planes in order to find (or bound) certain invariants of $X$.

When dim$(X) = 1$, one would like to degenerate the curve into a line arrangement with only nodes as singularities. This has been thoroughly investigated. For example, it is known that any smooth plane curve can be degenerated into a union of lines. However, the situation for a curve in $\mathbb{CP}^n$, $n > 2$ is completely different as there are, for example, smooth curves in $\mathbb{CP}^3$ which cannot be degenerated into a line arrangement with only double points (see [15]).

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