A quantitative characterisation of functions with low Aviles Giga energy on convex domains

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Abstract. Given a connected Lipschitz domain \( \Omega \) we let \( \Lambda(\Omega) \) be the set of functions in \( W^{2,2}(\Omega) \) with \( u = 0 \) on \( \partial \Omega \) and whose gradient (in the sense of trace) satisfies \( \nabla u(x) \cdot \eta_x = 1 \), where \( \eta_x \) is the inward pointing unit normal to \( \partial \Omega \) at \( x \).

The functional \( I_\epsilon(u) = \frac{1}{2} \int_\Omega \epsilon^{-1} \left| 1 - \left| \nabla u \right|^2 \right|^2 + \epsilon \left| \nabla^2 u \right|^2 \, dz \), minimised over \( \Lambda(\Omega) \), serves as a model in connection with problems in liquid crystals and thin film blisters. It is also the most natural higher order generalisation of the Modica and Mortola functional. In [16] Jabin, Otto and Perthame characterised a class of functions which includes all limits of sequences \( u_n \in \Lambda(\Omega) \) with \( I_{\epsilon_n}(u_n) \to 0 \) as \( \epsilon_n \to 0 \). A corollary to their work is that if there exists such a sequence \( (u_n) \) for a bounded domain \( \Omega \), then \( \Omega \) must be a ball and (up to change of sign) \( u := \lim_{n \to \infty} u_n \) is equal \( \text{dist}(\cdot, \partial \Omega) \). We prove a quantitative generalisation of this corollary for the class of bounded convex sets. Namely we show that there exists a positive constant \( \gamma_1 \) such that, if \( \Omega \) is a convex set of diameter 2 and \( u \in \Lambda(\Omega) \) with \( I_\epsilon(u) = \beta \), then \( |B_1(x) \triangle \Omega| \leq c\beta^{\gamma_1} \) for some \( x \) and

\[
\int_{\Omega} \left| \nabla u(z) + \frac{z - x}{|z - x|} \right|^2 \, dz \leq c\beta^{\gamma_1}.
\]

A corollary of this result is that there exists a positive constant \( \gamma_2 < \gamma_1 \) such that if \( \Omega \) is convex with diameter 2 and \( C^2 \) boundary with curvature bounded by \( \epsilon^{-1/2} \), then for any minimiser \( u \) of \( I_\epsilon \) over \( \Lambda(\Omega) \) we have

\[
\| u - \xi \|_{W^{1,2}(\Omega)} \leq c(\epsilon + \inf_y |\Omega \triangle B_1(y)|)^{\gamma_2},
\]

where \( \xi(z) = \text{dist}(z, \partial \Omega) \). Neither of the constants \( \gamma_1 \) or \( \gamma_2 \) are optimal.

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1. Introduction

We consider the following functional

\[
I_\epsilon(u) = \frac{1}{2} \int_{\Omega} \epsilon^{-1} \left| 1 - \left| \nabla u \right|^2 \right|^2 + \epsilon \left| \nabla^2 u \right|^2 \, dz
\]

the study of which arises from a number of sources, one of the earliest and most important of which is the article by Aviles and Giga [7]. We will refer to the quantity

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