A diameter bound for Sasaki manifolds

YASUFUMI NITTA

Abstract. In this paper we shall show that the diameter of a complete Sasaki manifold whose transverse Ricci curvature is bounded from below by a positive constant has a universal upper bound. This gives another proof of the result of Hasegawa and Seino in [9] which asserts that such manifolds are always compact with finite fundamental group.

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1. Introduction

A Sasaki manifold is a Riemannian manifold $(S, g)$ whose cone metric $\tilde{g} = dr^2 + r^2 g$ on $C(S) = S \times \mathbb{R}_+$ is Kähler. Then Sasakian geometry sits naturally in two aspects of Kähler geometry. For one thing, $(S, g)$ is the base of the cone manifold $(C(S), \tilde{g})$ which is a Kähler manifold. For another thing, any Sasaki manifold has a contact structure $(D, \eta, \xi)$, and it also has a 1-dimensional foliation $\mathcal{F}_\xi$, called the Reeb foliation, which admits a transverse Kähler metric $g^T$. Here, the Killing vector field $\xi$ is called the characteristic or Reeb vector field, the 1-form $\eta$ is called the contact 1-form, and the subbundle $D = \ker \eta$ is called the contact distribution (cf. Section 2). For this reason, Sasaki manifolds are often described as odd dimensional counterparts of Kähler manifolds. Then it is a natural problem to investigate how restrictions on the curvature for the “Kähler structure” influence the topology of the manifold. For instance, by a result of Harada [7], for any compact regular Sasaki manifold $(S, g)$ such that $h > \kappa^2$ and the lower bound of the sectional curvature is $1/\kappa^2$, the fundamental group $\pi_1(S)$ is cyclic. Here $h = \inf \{ H(X) ; X \in TS, \ |X| = 1 \}$ and $H(X)$ is the $\Phi$-holomorphic sectional curvature of $X$. (For the definition of $\Phi$-holomorphic sectional curvature, see [3].) Moreover, if additionally $S$ has minimal diameter $\pi$ then $S$ is isometric to the standard sphere (cf. [8]). The classification of complete simply connected Sasaki manifolds with constant $\Phi$-holomorphic sectional curvature is obtained by

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