Dimensionality and the stability of the Brunn-Minkowski inequality

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Abstract. We prove stability estimates for the Brunn-Minkowski inequality for convex sets. As opposed to previous stability results, our estimates improve as the dimension grows. In particular, we obtain a non-trivial conclusion for high dimensions already when

$$\text{Vol}_n \left( \frac{K + T}{2} \right) \leq 5 \text{Vol}_n(K) \text{Vol}_n(T).$$

Our results are equivalent to a thin shell bound, which is one of the central ingredients in the proof of the central limit theorem for convex sets.

1. Introduction

The Brunn-Minkowski inequality states, in one of its normalizations, that

$$\text{Vol}_n \left( \frac{K + T}{2} \right) \geq \sqrt{\text{Vol}_n(K) \text{Vol}_n(T)} \tag{1.1}$$

for any compact sets $K, T \subset \mathbb{R}^n$, where $(K + T)/2 = \{ (x + y)/2 : x \in K, y \in T \}$ is half of the Minkowski sum of $K$ and $T$, and where $\text{Vol}_n$ stands for the Lebesgue measure in $\mathbb{R}^n$. Equality in (1.1) holds if and only if $K$ is a translate of $T$ and both are convex, up to a set of measure zero.

The literature contains various stability estimates for the Brunn-Minkowski inequality, which imply that when there is almost-equality in (1.1), then $K$ and $T$ are almost-translates of each other. Such estimates appear in Diskant [8], in Groemer [13], and in Figalli, Maggi and Pratelli [11, 12]. We recommend Osserman [20] for a general survey on the stability of geometric inequalities.

All of the stability results that we found in the literature share a common feature: Their estimates deteriorate quickly as the dimension increases. For instance, suppose that $K, T \subset \mathbb{R}^n$ are convex sets with

$$\text{Vol}_n(K) = \text{Vol}_n(T) = 1 \quad \text{and} \quad \text{Vol}_n \left( \frac{K + T}{2} \right) \leq 5. \tag{1.2}$$

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