Some remarks on Thom’s transversality theorem

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Abstract. We study Thom’s transversality theorem using a point of view, suggested by Gromov, which allows to avoid the use of Sard’s theorem and gives finer information on the structure of the set of non-transverse maps.

Mathematics Subject Classification (2010): 58B15 (primary); 58A35, 58D20 (secondary).

The smooth image of \( \mathbb{R}^n \) in \( \mathbb{R}^d \), for \( d > n \) is rectifiable of positive codimension, and therefore it has zero measure. This can be seen as the easy case of Sard’s Theorem, which states that the set of critical values of a smooth map has zero measure. The most common proofs of the Thom transversality theorem rely on the general form of Sard’s theorem. Our goal in the present paper is to develop an approach, suggested by Gromov, in [5, page 33], and used in [8, Section 2.3] to this theorem using only the easy case of Sard’s Theorem, which leads to a stronger form of Thom’s theorem. The key technical tool in this approach is our Conjecture 3.1 below. The main novelty in the present paper is the proof of several cases of that conjecture, Theorems 3.2 and 3.3 in Section 3. These special cases are enough to derive new variants of the Thom transversality theorem, stated as Theorem 1.4 in the introduction. We work in a context closely inspired from [1], and we use a notion of rectifiable sets in Banach spaces coming from [14] and [4], as recalled in Section 2.

1. Introduction

It is well-known that “most” functions are Morse, which means that their critical points are non-degenerate. Discussing this claim with some details will be an occasion to introduce and motivate the present work. Let us fix some integer \( r \geq 2 \) and a dimension \( n \). Let \( B^n \) be the open unit ball in \( \mathbb{R}^n \), and \( \bar{B}^n \) the closed unit ball. We denote by \( C^r(\bar{B}^n, \mathbb{R}) \) the space of functions which are \( C^r \) on \( B^n \), and whose differentials up to order \( r \) extend by continuity to the closed ball \( \bar{B}^n \). We endow this space with the norm given by the sum of the suprema of the differentials of order less than \( r \). It is then a separable Banach space. Let \( F \) be an affine subspace