Degree growth of birational maps of the plane

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Abstract. This article studies the sequence of iterative degrees of a birational map of the plane. This sequence is known either to be bounded or to have a linear, quadratic or exponential growth.

The classification elements of infinite order with a bounded sequence of degrees is achieved, the case of elements of finite order being already known. The coefficients of the linear and quadratic growth are then described, and related to geometrical properties of the map. The dynamical number of base-points is also studied.

Applications of our results are the description of embeddings of the Baumslag-Solitar groups and GL(2, ℚ) into the Cremona group.

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1. Introduction

A rational map of the complex projective plane \( \mathbb{P}^2 = \mathbb{P}^2_\mathbb{C} \) into itself is a map of the following type

\[ \phi: \mathbb{P}^2 \rightarrow \mathbb{P}^2, \quad (x : y : z) \rightarrow (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)), \]

where the \( \phi_i \)’s are homogeneous polynomials of the same degree without common factor. The degree \( \deg \phi \) of \( \phi \) is by definition the degree of these polynomials. We will only consider birational maps, which are rational maps having a rational inverse, and denote by Bir(\( \mathbb{P}^2 \)) the group of such maps, classically called Cremona group.

We are interested in the behaviour of the sequence \( \{\deg \phi^k\}_{k \in \mathbb{N}} \). According to [13], the sequence is either bounded or has a linear, quadratic or exponential growth. We will say that \( \phi \) is

1. elliptic if the growth is bounded;
2. a Jonquières twist if the growth is linear;

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