On Harnack inequalities and optimal transportation

DOMINIQUE BAKRY, IVAN GENTIL AND MICHEL LEDOUX

Abstract. We develop connections between Harnack inequalities for the heat flow of diffusion operators with curvature bounded from below and optimal transportation. Through heat kernel inequalities, a new isoperimetric-type Harnack inequality is emphasized. Commutation properties between the heat and Hopf-Lax semigroups are developed consequently, providing direct access to heat flow contraction in Wasserstein spaces.

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1. Introduction

Harnack inequalities classically provide strong tools towards regularity properties of solutions of partial differential equations and heat kernel bounds. A renowned result on the topic is the parabolic inequality by P. Li and S.-T. Yau [29]

$$\frac{|\nabla P_t f|^2}{(P_t f)^2} - \frac{\Delta P_t f}{P_t f} \leq \frac{n}{2t}$$

(1.1)

for the heat semigroup \((P_t)_{t \geq 0}\) on an \(n\)-dimensional Riemannian manifold \((M, g)\) with non-negative Ricci curvature, and every \(t > 0\) and positive (measurable) function \(f : M \to \mathbb{R}\). By integration along geodesics, it yields the Harnack inequality

$$P_t f(x) \leq P_{t+s} f(y) \left( \frac{t+s}{t} \right)^{n/2} \exp^{d(x,y)^2/4s}$$

(1.2)

for \(f : M \to \mathbb{R}\) non-negative and \(t, s > 0\), where \(d(x, y)\) is the Riemannian distance between \(x, y \in M\). The results (1.1) and (1.2) admit versions for any lower bound on the Ricci curvature \((cf. [18,29])\). A heat flow proof of (1.1), in the spirit of the arguments developed in this work, has been provided in [10].

In the context of diffusion operators, the Harnack inequality (1.2) may actually loose its relevance due to the infinite-dimensional feature of some models. Let

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