A perturbation result for the Riesz transform

BAPTISTE DEVVYVER

Abstract. We show a perturbation result for the Riesz transform: if $M_0$ and $M_1$ are complete Riemannian manifolds which are isometric outside a compact set, we give sufficient conditions so that the boundedness on $L^p$ of the Riesz transform on $M_0$ implies the boundedness on $L^p$ of the Riesz transform on $M_1$.

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1. Introduction

Let $(M, g)$ be a Riemannian manifold. The Riesz transform problem, namely giving conditions on $p$ and on the manifold such that the operator $d\Delta^{-1/2}$ – the so-called Riesz transform – is bounded on $L^p$, has recently undergone certain progress. A pioneering result which goes back to 1985 is a theorem of D. Bakry [2] which asserts that if the Ricci curvature of $M$ is non-negative, then the Riesz transform on $M$ is bounded on $L^p$ for every $1 < p < \infty$. However, it is only recently that some progresses have been made to understand the behaviour of the Riesz transform if some amount of negative Ricci curvature is allowed. A general question is the following:

**Question 1.1.** What is the analogue of Bakry’s result for manifolds with some (small) amount of negative Ricci curvature?

Here, the smallest of the negative part of the Ricci curvature $\text{Ric}_-$ should be understood in an integral sense, i.e. $\text{Ric}_- \in L'(d\mu)$, for some value of $\mu$ and some measure $d\mu$. A partial answer has been provided by T. Coulhon and Q. Zhang in [11], where it is shown essentially that if the negative part of the Ricci curvature is smaller in an integral sense than a constant $\varepsilon$ (depending on the geometry of the manifold under consideration), then the Riesz transform is bounded on $L^p$ for every $1 < p < \infty$. However, this result is not entirely satisfying, since it does not say what happens if the integral of the Ricci curvature is bigger than the threshold $\varepsilon$: thus, it does not cover the case of manifolds having non-negative Ricci curvature outside a compact set. Unlike manifolds with non-negative Ricci curvature, man-

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